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CELLULAR AUTOMATA CAN GENERATE FRACTALS

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Let L be the transition rule of a cellular automaton which is linear modulo 2. Associated to L there is defined a compact subspace of Euclidean space related to the behavior of L under iteration. It is seen that this subspace can have fractional Hausdorff dimension.

1. Introduction

Let L be the transition rule of a cellular automaton and let ω be a configuration. Thus ω may be considered an assignment of a zero or a one to each square of an infinite n-dimensional checkerboard; and L assigns to ω a new configuration $L\omega$ in a systematic way. (More details are given in Section 2.) Our general problem is to study the sequence $\omega, L\omega, L^2\omega = L(L\omega), L^3\omega, \ldots$.

In this paper we define an invariant of L in the case where L is 'linear'. We call this invariant $\lim L$, and it is a compact subset of $\mathbb{R}^{n+1}$. It turns out that $\lim L$ can be a 'fractal'; i.e., its Hausdorff dimension need not be integral. Thus for Example 4.2 $\lim L$ has dimension $\log_2 3$, while for Example 4.3 it has dimension $\log_2(1+\sqrt{5})$. The author finds it intriguing that such numbers arise as invariants of such simple cellular automata. Example 4.2 is merely a mod 2 analogue of Pascal's triangle.

The definition of $\lim L$ is somewhat complicated. Essentially, if ω is any finite nonzero configuration, one can consider, for any p, putting a copy of ω below a copy of $L\omega$ below a copy of $L^2\omega$ below a copy of $L^3\omega \ldots$ below a copy of $L^{2^p}\omega$. Thus one obtains a partial 'graph' of L, called $F^{2^p}\omega$. Consider this as a subset of $\mathbb{R}^{n+1}$ by identifying a 'square' to which 1 has been assigned with the lattice vector indexing the square. Rescale by division by the scalar 2^p, yielding $F^{2^p}\omega/2^p$. Then $\lim L$ is the set of limit points of the sets $F^{2^p}\omega/2^p$ for $p=1,2,\ldots$. It turns out, surprisingly, that $\lim L$ is independent of the choice of ω, for ω finite and nonzero. It describes, roughly, the appearance of $F^{2^p}\omega$, by $F^{2^p}\omega \approx 2^p \cdot \lim L$ for large p. Thus the invariance of $\lim L$ indicates that the gross appearance of $F^{2^p}\omega$ is independent of ω, as long as ω is finite and nonzero and p is large.

The prominence of powers of 2 may be explained by the fact that L is assumed linear in a sense using arithmetic modulo 2. Arithmetic mod 2 is natural since squares are assigned only zeroes or ones.

This paper continues the method of Willson [10] to study cellular automata via certain geometric invariants. Unlike the other paper, however, this one applies to linear automata. Linear automata have been studied in Amoroso and Cooper [1], Ostrand [7], Barto [2]. A general method of constructing limits possibly of fractional dimension has recently been investigated by Dekking [3]; the approach here differs in that we make essential use of the cellular automata structure while Dekking considers operations on finite sequences of symbols and a geometric realization of these.

In Section 2 we present the basic notions and define $\lim L$. In Section 3 we prove the invariance of $\lim L$. In Section 4 we consider the Hausdorff dimension of $\lim L$ and present some examples.

2. Basic notions

Let n be a fixed positive integer; $\mathbb{R}^n$ is Euclidean space of dimension n; $\mathbb{Z}^n \subset \mathbb{R}^n$ is the set of points all of whose coordinates are integers; $\mathbb{Z}/2$ denotes the integers mod 2.

Definitions. A *configuration* ω on $\mathbb{Z}^n$ is a map $\omega : \mathbb{Z}^n \to \mathbb{Z}/2$. We denote the set of all configurations on $\mathbb{Z}^n$ by $\mathscr{P}^n$. A configuration ω is *finite* provided $\omega(v) = 1$ for only finitely many v. The configuration 0 assigns 0 to all $v \in \mathbb{Z}^n$. The configuration δ is defined by $\delta(v) = 1$ if $v = (0, 0, \dots, 0)$ and $\delta(v) = 0$ otherwise. We define two kinds of addition: If $\omega, \tau \in \mathscr{P}^n$ we may define $\omega + \tau \in \mathscr{P}^n$ by $(\omega + \tau)(v) = \omega(v) + \tau(v) \bmod 2$ for $v \in \mathbb{Z}^n$. If $\omega \in \mathscr{P}^n$ and $v \in \mathbb{Z}^n$, we may define the *translate* of ω by v as $\omega + v$ where $(\omega + v)(w) = \omega(w - v)$.

To each configuration ω we associate a subset of $\mathbb{Z}^n \subset \mathbb{R}^n$ by saying $\omega(v) = 1$ if and only if v lies in that subset. It turns out that there is no confusion in writing ω for that subset of $\mathbb{Z}^n$. Thus $\omega(v) = 1$ iff $v \in \omega$; and a configuration ω may be interpreted geometrically as a subset ω of $\mathbb{Z}^n$. Equivalently, ω is an assignment of zero or one to each square of an n-dimensional checkerboard; and ω is also regarded as the set of squares to which the value one has been assigned.

Definition. A *transition rule* F on $\mathscr{P}^n$ is a map $F : \mathscr{P}^n \to \mathscr{P}^n$ such that (1) $F(0) = 0$; and (2) there exist $v_1, \dots, v_m \in \mathbb{Z}^n$ and a map $f : (\mathbb{Z}/2)^m \to \mathbb{Z}/2$ [where $(\mathbb{Z}/2)^m$ denotes the m-tuples of integers mod 2] such that $(F\omega)(v) = f(\omega(v + v_1), \dots, \omega(v + v_m))$ for all $v \in \mathbb{Z}^n$, $\omega \in \mathscr{P}^n$. We call f the *generating function* for F. The iterates F^q are defined by $F^0\omega = \omega$, $F^1\omega = F\omega$, $F^2\omega = F(F\omega)$, $F^3\omega = F(F^2\omega)$, etc. Note that F is necessarily invariant under translation in the sense that $F(\omega + v) = (F\omega) + v$ for $\omega \in \mathscr{P}^n$, $v \in \mathbb{Z}^n$.

Definition. The transition rule F on $\mathscr{P}^n$ is *linear* provided its generating function f is linear; or equivalently, provided $F(\omega + \tau) = F\omega + F\tau$ for $\omega, \tau \in \mathscr{P}^n$. Here we use arithmetic mod 2.

Linear transition rules have been studied by Barto [2], Ostrand [7], and Amoroso and Cooper [1]. Among their properties are the following, which are easy to verify:

Proposition 2.1. *Let $L : \mathscr{P}^n \to \mathscr{P}^n$ be a linear transition rule. Then the following are true:*

(1) *For any positive integer q, L^q is linear.*
(2) *If $v \in L\delta$, then $2v \in L^2\delta$; conversely if $w \in L^2\delta$, then $\frac{1}{2}w \in L\delta$.*
(3) *More generally, $v \in L\delta$ iff $2^q v \in L^{2^q}\delta$.*
(4) If *$\omega \in \mathscr{P}^n$, then $L\omega = \sum_{w \in \omega} L(\delta + w)$; hence $L\omega \subseteq \bigcup_{w \in \omega} L(\delta + w)$.*

We note that in (2) and (3), the 2 and $\frac{1}{2}$ refer to scalar multiplication; thus $2v = v + v \in \mathbb{R}^n$.

Definition. Let L be a transition rule on $\mathscr{P}^n$. We define a transition rule F on $\mathscr{P}^{n+1}$, called the *graph* of L, as follows: Write elements of $\mathbb{R}^{n+1}$ as (v, r) where $v \in \mathbb{R}^n$, $r \in \mathbb{R}$. Let $l : (\mathbb{Z}/2)^m \to \mathbb{Z}/2$ be the generating function for L, so

$$(L\omega)(v) = l(\omega(v + v_1), \ldots, \omega(v + v_m)) \quad \text{when } \omega \in \mathscr{P}^n.$$

Now define $F\tau$, for $\tau \in \mathscr{P}^{n+1}$, by setting $(F\tau)(v, r) = 1$ if $\tau(v, r) = 1$; while if $\tau(v, r) = 0$, set

$$(F\tau)(v, r) = l(\tau(v + v_1, r - 1), \tau(v + v_2, r - 1), \ldots, \tau(v + v_m, r - 1)).$$

Let F be the graph of L. The natural embedding of $\mathbb{Z}^n$ in $\mathbb{Z}^{n+1}$ takes $(a_1, \ldots, a_n)$ to $(a_1, \ldots, a_n, 0)$; this induces an inclusion $\mathscr{P}^n \subset \mathscr{P}^{n+1}$, by regarding $\omega \in \mathscr{P}^n$ as a subset of $\mathbb{Z}^n$, hence a subset of $\mathbb{Z}^{n+1}$ via the natural embedding, hence an element of $\mathscr{P}^{n+1}$. Thus $F^p\omega$ is defined if $\omega \in \mathscr{P}^n$. For any $\omega \in \mathscr{P}^n$, for any positive integer p, and for any integer q so $0 \le q \le p$, we see that $(v, q) \in F^p\omega$ iff $v \in L^q\omega$. This equivalence is the basic property of F and explains the use of the term 'graph'.

In order to define our invariant, we need the notion of Kuratowski limits: (See Kuratowski [5; pp. 241–250] or Salinetti and Wets [9].)

Definition. Let $X_1, X_2, \ldots$ be a sequence of subsets of $\mathbb{R}^n$. Define

$\liminf X_p = \{x \in \mathbb{R}^n$: there exists a sequence $x_p \in X_p$ for $p = 1, 2, \ldots$ such that $\lim_{p\to\infty} x_p = x\}$.

$\limsup X_p = \{x \in \mathbb{R}^n$: there exists a sequence of positive integers p_i so $p_i \to \infty$ as $i \to \infty$ and a sequence $x_i \in X_{p_i}$ such that $\lim_{i\to\infty} x_i = x\}$.

Clearly $\liminf X_p \subseteq \limsup X_p$. If they are equal we denote them both by $\lim X_p$. It is not hard to show that each is a closed subset of $\mathbb{R}^n$.

We can now define the invariant which is the main object of study in this paper:

Definition. Let $L:\mathscr{P}^n\to\mathscr{P}^n$ be a linear transition rule. Let F be the graph of L and let $\omega\in\mathscr{P}^n$. Define $\liminf(L,\omega)$ to be $\liminf X_p$ and define $\limsup(L,\omega)$ to be $\limsup X_p$ where $X_p=(1/2^p)F^{2^p}\omega\subset\mathbb{R}^{n+1}$. Here $F^{2^p}\omega$ is regarded as a subset of $\mathbb{R}^{n+1}$, and each vector in $F^{2^p}\omega$ is being multiplied by the scalar $(1/2^p)$.

Clearly, if ω is finite, it follows $\liminf(L,\omega)$ and $\limsup(L,\omega)$ are compact subsets of $\mathbb{R}^{n+1}$. It will be proved in Section 3 that $\liminf(L,\omega)=\limsup(L,\omega)$ if ω is finite and nonzero; moreover this subset of $\mathbb{R}^{n+1}$ will be independent of ω. Thus there exists a compact subspace of $\mathbb{R}^{n+1}$, which we call $\lim L$, such that $\lim L=\liminf(L,\omega)=\limsup(L,\omega)$ for any finite nonzero ω.

3. Invariance of $\lim L$

This section is devoted to the proof of the following theorem: Throughout the section, $L:\mathscr{P}^n\to\mathscr{P}^n$ is linear, $F:\mathscr{P}^{n+1}\to\mathscr{P}^{n+1}$ is the graph of L, and $\mathbb{Z}^n$ is regarded a subset of $\mathbb{Z}^{n+1}$ by the natural inclusion (whence $\mathscr{P}^n\subset\mathscr{P}^{n+1}$).

Theorem 3.1. *Suppose $\omega\in\mathscr{P}^n$ is finite and nonzero. Then*

$$\liminf(L,\omega)=\limsup(L,\omega)=\liminf(L,\delta)=\limsup(L,\delta).$$

We call this common subset of $\mathbb{R}^{n+1}$ by the name $\lim L$. The theorem implies that $\lim L$ depends only on L. It is essential in Theorem 3.1 that $\omega\in\mathscr{P}^n$ and not merely $\mathscr{P}^{n+1}$. Thus in Example 4.4 we give a finite nonzero $\omega\in\mathscr{P}^{n+1}$ so that $\liminf(1/2^p)F^{2^p}\omega\neq\lim L$.

The proof of Theorem 3.1 will occupy a sequence of lemmas.

Lemma 3.2. *If $v\in\mathbb{Z}^n$, then $\liminf(L,\omega+v)=\liminf(L,\omega)$. Thus $\liminf(L,\omega)$ is invariant under translation of ω.*

Proof. Since F is invariant under translation,

$$\begin{aligned}\liminf(L,\omega+v)&=\liminf F^{2^p}(\omega+v)/2^p=\liminf((F^{2^p}\omega)+v)/2^p\\&=\liminf F^{2^p}\omega/2^p=\liminf(L,\omega).\qquad\square\end{aligned}$$

Lemma 3.3. *Suppose $\omega\in\mathscr{P}^n$ is finite and nonzero. Then*

$$\liminf(L,\omega)=\liminf(L,\delta).$$

Proof. By Lemma 3.2 we may assume $0\in\omega$. We first prove that $\liminf(L,\delta)\subseteq\liminf(L,\omega)$. Suppose $(x,r)\in\liminf(L,\delta)$, where $x\in\mathbb{R}^n$, $r\in\mathbb{R}$. Then there exists a sequence $(x_p, r_p) \in F^{2^p}\delta$ so that $(x_p, r_p)/2^p \to (x, r)$ as $p \to \infty$. Since F is the graph of L, $x_p \in L^{r_p}\delta$. Since L is linear, for any a we see $2^a x_p \in L^{2^a r_p}\delta$ by Proposition 2.1. Choose a so large that $2^a > 2\Delta(\omega)$ [where $\Delta(\omega)$ is the diameter of ω]. Since the elements of $L^{r_p}\delta$ have distance at least 1 from each other, it follows from Proposition 2.1 that the elements of $L^{2^a r_p}\delta$ have distance at least 2^a from each other. Hence the sets $\omega + v$, where v ranges over the elements of $L^{2^a r_p}\delta$, are pairwise disjoint. Since

$$L^{2^a r_p}\omega = L^{2^a r_p}\Big(\sum_{w\in\omega}(\delta + w)\Big) = \sum_{w\in\omega}(L^{2^a r_p}\delta + w)$$

$$= \sum_{w\in\omega}\ \sum_{v\in L^{2^a r_p}\delta}(\delta + v + w) = \sum_{v\in L^{2^a r_p}\delta}(\omega + v)$$

and the latter sets are pairwise disjoint, we conclude

$$L^{2^a r_p}\omega = \bigcup_{v\in L^{2^a r_p}\delta}(\omega + v).$$

Since $2^a x_p \in L^{2^a r_p}\delta$ and $0 \in \omega$, it follows $2^a x_p \in L^{2^a r_p}\omega$, whence $(2^a x_p, 2^a r_p) \in F^{2^a 2^p}\omega$. Thus

$$\lim_{p\to\infty}(2^a x_p, 2^a r_p)/(2^a 2^p) = (x, r) \in \liminf(L, \omega)$$

and $\liminf(L, \delta) \subseteq \liminf(L, \omega)$.

We now prove that $\liminf(L, \omega) \subseteq \liminf(L, \delta)$. Suppose $(x, r) \in \liminf(L, \omega)$. Then there exists a sequence $(x_p, r_p) \in F^{2^p}\omega$ so $(x_p, r_p)/2^p \to (x, r)$. Then $x_p \in L^{r_p}\omega \subseteq \bigcup_{v\in\omega} L^{r_p}(\delta + v)$ by Proposition 2.1. Hence we may choose $v_p \in \omega$ so $x_p \in L^{r_p}(\delta + v_p)$, or equivalently $x_p - v_p \in L^{r_p}\delta$. Hence $(x_p - v_p, r_p) \in F^{2^p}\delta$ and

$$\lim(x_p - v_p, r_p)/2^p = (x, r) \in \liminf(L, \delta),$$

since the vectors v_p in ω are bounded. □

Lemma 3.4. *Suppose $\omega \in \mathscr{P}^n$ is finite and nonzero. Then*

$$\limsup(L, \omega) = \limsup(L, \delta).$$

Proof. Analogous to the proof of Lemma 3.3. □

Lemma 3.5. *Suppose $(x, q) \in F^{2^a}\delta$. Then $(x, q)/2^a \in \liminf(L, \delta)$.*

Proof. If $(x, q) \in F^{2^a}\delta$, then $x \in L^q\delta$, whence $2^p x \in L^{2^p q}\delta$ by Proposition 2.1. Thus $(2^p x, 2^p q) \in F^{2^p 2^a}\delta$ and

$$\lim_{p\to\infty}(2^p x, 2^p q)/(2^p 2^a) = (x, q)/2^a \in \liminf(L, \delta). \qquad \square$$

Lemma 3.6. $\liminf(L, \delta) = \limsup(L, \delta)$.

Proof. It is immediate that $\liminf(L,\delta) \subseteq \limsup(L,\delta)$. Conversely, suppose $(x,r) \in \limsup(L,\delta)$. Then there is a sequence $(x_i, r_i) \in F^{2^{(p_i)}}\delta$ so that $(x_i, r_i)/2^{(p_i)} \to (x,r)$ and $p_i \to \infty$ as $i \to \infty$. By Lemma 3.5, $(x_i, r_i)/2^{(p_i)} \in \liminf(L,\delta)$. Since $\liminf(L,\delta)$ is closed, it follows that $(x,r) \in \liminf(L,\delta)$. □

Theorem 3.1 follows directly from Lemmas 3.3, 3.4, and 3.6.

4. The Hausdorff dimension of lim L

In this section we show that if $L : \mathscr{P}^n \to \mathscr{P}^n$ is linear, then $\lim L \subset \mathbb{R}^{n+1}$ can have fractional Hausdorff dimension.

We shall use the notation of Hurewicz and Wallman [4; pp. 102–107]. (A much more detailed account is given in Rogers [8].) Suppose $X \subset \mathbb{R}^{n+1}$ and suppose $0 \le d < \infty$. Given $\varepsilon > 0$, we set $m_d^\varepsilon(X) = \inf \sum_{i=1}^\infty (\Delta(A_i))^d$ where Δ indicates diameter and $X \subseteq A_1 \cup A_2 \cup \cdots$ while $\Delta(A_i) < \varepsilon$ for all i. We call $\sum (\Delta(A_i))^d$ a *d-sum*. Define $m_d(X) = \sup_{\varepsilon > 0} m_d^\varepsilon(X)$. One can see that there is a unique D, $0 \le D \le n+1$ so $m_d(X) = \infty$ for $d < D$ and $m_d(X) = 0$ for $d > D$. This value D is called the *Hausdorff dimension* of X, and we write it $g \dim X$ (for 'geometric' dimension).

Our first result is a lemma for computing $g \lim x$ in a special situation. It is a special case of the folklore rule of thumb that 'similarity dimension' equals Hausdorff dimension (see Mandelbrot [6]).

Theorem 4.1. *Let $X \subset \mathbb{R}^n$ be compact.*

(i) *Suppose for some integer $a \ge 2$ and vectors $v_1, \dots, v_m$ we have $aX \subseteq \bigcup_{i=1}^m (X + v_i)$. Then $g \dim X \le \log_a m$.*

(ii) *Suppose for some integer $a \ge 2$ and for integer vectors $v_1, \dots, v_m$ which are pairwise distinct* mod a *we have $aX \supseteq \bigcup_{i=1}^m (X + v_i)$. Then $g \dim X \ge \log_a m$.*

(iii) *If $aX = \bigcup_{i=1}^m (X + v_i)$ where a and the v_i are as in* (ii), *then $g \dim X = \log_a m$.*

Proof. (i) We show that if $D > \log_a m$, then $m_D(X) = 0$. Let $\{C_i : i = 1, \dots, r\}$ be a finite cover of X with corresponding D-sum $\sum_{i=1}^r \Delta(C_i)^D = N$ and $\Delta(C_i) < K$ for all i. Since $X \subseteq \bigcup_{i=1}^m ((1/a)X + (v_i/a))$, we see

$$\{(1/a)C_i + (v_j/a) : i = 1, \dots, r;\, j = 1, \dots, m\}$$

is also a cover of X, each member of which has diameter $< (1/a)K$, and whose D-sum equals

$$\sum_{i,j} \Delta((1/a)C_i + (v_j/a))^D = \sum_{i,j} (1/a)^D \Delta(C_i)^D = (m/a^D)N.$$

By iterating this argument, we see that for each p, X has a cover, each element of which has diameter less than $(1/a)^p K$ and whose D-sum equals $(m/a^D)^p N$. Since $D > \log_a m$, and $a \ge 2$, it easily follows that $m_D(X) = 0$.

We conclude $g\dim X \le \log_a m$.

(ii) We suppose $D < \log_a m$ and prove $m_D(X) > 0$. It will then follow that $g\dim X \ge \log_a m$ and the proof will be complete.

Suppose $\{C_i : i = 1, \dots, r\}$ is a cover of X satisfying $\Delta(C_i) < \varepsilon$ for each i, with D-sum $\sum \Delta(C_i)^D = N$; we show N is bounded away from 0. Choose a positive number K such that any set of diameter less than K meets $X + \lambda_i$ for at most B different i, where λ_i ranges over all integer vectors of $\mathbb{R}^n$. For any positive integer p consider the sets $\{a^p C_i : i = 1, \dots, r\}$, a cover of $a^p X$. Note

$$a^2 X \supseteq \bigcup a(X + v_i) = \bigcup (aX + av_i) \supseteq \bigcup (X + v_j + av_i);$$

more generally,

$$a^p X \supseteq \bigcup (X + v_{i_1} + av_{i_2} + a^2 v_{i_3} + \cdots + a^{p-1} v_{i_p}).$$

Since the vectors v_i are distinct mod a, it is easy to see that the vectors $v_{i_1} + av_{i_2} + \cdots + a^{p-1} v_{i_p}$ are all distinct, hence m^p in number.

Choose a positive integer q so that $(a^D/m)^q \le (1/B)$. We show that $N \ge (K/a^q)^D$ proving that N is bounded away from 0. Two cases arise: Either (1) the maximal diameter of all C_i is less than K/a^q; or else (2) that diameter is $\ge K/a^q$. In case (2), we see $N \ge (K/a^q)^D$ directly. In case (1), we observe that by the definition of K, each set $a^q C_i$ meets at most B of the sets $X + v_{i_1} + av_{i_2} + \cdots + a^{q-1} v_{i_q}$. Let

$$A(i_1, \dots, i_q) = \{ j : a^q C_j \text{ meets } X + v_{i_1} + av_{i_2} + \cdots + a^{q-1} v_{i_q} \}.$$

Then

$$\sum_{(i_1, \dots, i_q)} \sum_{j \in A(i_1, \dots, i_q)} \Delta(a^q C_j)^D \le B a^{qD} N.$$

Since there are precisely m^q such $(i_1, \dots, i_q)$, there exists at least one $(i_1, \dots, i_q)$ so that

$$\sum_{j \in A(i_1, \dots, i_q)} \Delta(a^q C_j)^D \le B(a^D/m)^q N \le N,$$

the last inequality by the choice of q. Thus, regarding $X + v_{i_1} + \cdots + a^{q-1} v_{i_q}$ as another copy of X, we see that, in case (1) X has a new cover (via translates of certain $a^q C_i$) with D-sum $\le N$, and for which the minimal diameter of an element is at least a^q times the minimal diameter of an element of the original cover. This new cover falls into either case (1) or case (2), and we can repeat the argument. Since the original cover was finite ultimately we find a cover satisfying case (2), whence $N \ge (K/a^q)^D$. Thus $m_D(X) \ge (K/a^q)^D > 0$ and the proof of (ii) is complete.

Part (iii) is an immediate consequence. □

Example 4.2. Let $n = 1$. Define $L : \mathscr{P}^1 \to \mathscr{P}^1$ by $(L\omega)(v) = \omega(v) + \omega(v-1) \bmod 2$, for $v \in \mathbb{Z}^1$. Then L is linear and $g\dim(\lim L) = \log_2 3$.

Proof. We shall only sketch the proof. Let $X = \lim L$. Then clearly $X \subset \{(x, r) : 0 \le r \le 1\}$. We shall see that X equals the union of $\frac{1}{2}X$, $\frac{1}{2}X + (0, \frac{1}{2})$, $\frac{1}{2}X + (\frac{1}{2}, \frac{1}{2})$. Hence

$2X = X \cup (X + (0,1)) \cup (X + (1,1))$. Theorem 4.1 then applies to yield $g \lim X = \log_2 3$.

First, suppose $(x,r) \in X$. Then for some $(x_p, r_p) \in F^{2^p}\delta$, $(x_p, r_p)/2^p \to (x,r)$. But $(x_p, r_p) \in F^{2^{p+1}}\delta$ also, so $(x_p, r_p)/2^{p-1} \to (x,r)/2 \in X$. Thus $\frac{1}{2}X \subseteq X$. Conversely, suppose $(x,r) \in X$ and $0 \le r \le \frac{1}{2}$. Then for (x_p, r_p) as above, for large $p, r_p < (\frac{1}{2})2^p$, whence

$$(x_p, r_p) \in F^{2^{p-1}}\delta \quad \text{and} \quad \lim_{p\to\infty} (x_p, r_p)/2^{p-1} = (2x, 2r) \in X.$$

Thus $X \cap \{(x,r) : r < \frac{1}{2}\} \subseteq \frac{1}{2}X$. A direct argument shows that the closed line segment from $(0, \frac{1}{2})$ to $(\frac{1}{2}, \frac{1}{2})$ lies in X and hence $X \cap \{(x,r) : r \le \frac{1}{2}\} = \frac{1}{2}X$.

Since $L^{2^p}\delta$ assigns 1 to only 0 and 2^p it is easy to see that $F^{2^{p+1}}\delta$ assigns 1 only to integer vectors of three types:

(i) (x,r) with $0 \le r \le 2^p$, $x \in L^r\delta$;

(ii) (y,r) of form $(y,r) = (x,r) + (0, 2^p)$ where (x,r) has type (i);

(iii) (y,r) of form $(y,r) = (x,r) + (2^p, 2^p)$ where (x,r) has type (i).

As we let $p \to \infty$, it follows that

$$X = \tfrac{1}{2}X \cup (\tfrac{1}{2}X + (0, \tfrac{1}{2})) \cup (\tfrac{1}{2}X + (\tfrac{1}{2}, \tfrac{1}{2})).$$

This completes the argument that $g \dim(\lim L) = \log_2 3$.

We remark that $F^{2^p}\delta$ resembles a mod 2 version of Pascal's triangle.

Example 4.3. Let $n = 1$. Define $L : \mathscr{P}^1 \to \mathscr{P}^1$ by

$$(L\omega)(v) = \omega(v) + \omega(v-1) + \omega(v-2) \bmod 2, \quad \text{for } v \in \mathbb{Z}.$$

Then L is linear and $g \dim(\lim L) = \log_2(1 + \sqrt{5})$.

The proof resembles that of Example 4.2 but is harder and requires a more elaborate version of Theorem 4.1. We omit it.

Example 4.4. Let $L : \mathscr{P}^1 \to \mathscr{P}^1$ be as in Example 4.3, let F be its graph, and let $\omega \in \mathscr{P}^2$ assign one only to $(0,0)$ and $(-1,-1)$. Then it is easy to see that $\lim(1/2^p)F^{2^p}\omega$ is the convex hull of the points $(0,0)$, $(0,1)$, and $(+2,1)$. Hence $\lim(1/2^p)F^{2^p}\omega \ne \lim L$. We conclude that in Theorem 3.1 it is essential that $\omega \in \mathscr{P}^n$ and not merely $\omega \in \mathscr{P}^{n+1}$.

We conclude with a result showing that $g \dim(\lim L)$ is a weaker invariant than is $\lim L$.

Theorem 4.5. *Let q be a positive integer, $L : \mathscr{P}^n \to \mathscr{P}^n$ be linear. Then*

$$g \dim(\lim L) = g \dim(\lim L^q).$$

Proof. Let F be the graph of L, G the graph of L^q, $X = \lim L$, $Y = \lim L^q$. Suppose $(x,r) \in X$. We shall show that $(x, r/q) \in Y$. Since the map $\phi : (x,r) \to (x, r/q)$ is a linear homeomorphism of X onto a subset of Y, it will easily follow that $g\dim X = g\dim \phi(X) \leq g\dim Y$. To see that $(x, r/q) \in Y$, choose $(x_p, r_p) \in F^{2^p}\delta$ so that $(x_p, r_p)/2^p \to (x, r)$. Choose m_p so that m_p is divisible by q and $r_p - q < m_p \leq r_p$. There exists C independent of p and $t_p \in L^{m_p}\delta$ so that $\|x_p - t_p\| \leq C$. Hence $(t_p, m_p/q) \in G^{2^p}\delta$, and thus

$$\lim_{p\to\infty} (t_p, m_p/q)/2^p = (x, r/q) \in Y.$$

Now we suppose $(y, r) \in Y$ and we choose s so $q \leq 2^s$. We shall show that $(y/2^s, qr/2^s) \in X$. Since the map $\psi : (y, r) \to (y/2^s, qr/2^s)$ is a linear homeomorphism of Y onto a subset of X, it will follow $g\dim Y \leq g\dim X$. This will complete the proof.

So suppose $(y_p, r_p) \in G^{2^p}\delta$ satisfies $(y_p, r_p)/2^p \to (y, r)$. Then $y_p \in L^{qr_p}\delta$, so $(y_p, qr_p) \in F^{2^{s+p}}\delta$ and $\lim(y_p, qr_p)/2^{s+p} = (y/2^s, qr/2^s) \in X$. $\square$

Remark. If L is as in Example 4.2, then $g\dim(\lim L^3) = g\dim(\lim L) = \log_2 3$ by Theorem 4.5. It is obvious, however, that $\lim L^3 \neq \lim L$, since, for example, $(3,1) \in \lim L^3$ via Lemma 3.4, while $(3,1) \notin \lim L$.

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